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Opening of a gap in an inhomogeneous external field

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Abstract. We study the one-dimensional spin-1/2 antiferromagnetic Heisenberg model exposed to an external field, which is a superposition of a homogeneous field h_3 and a small periodic field of strength h_1. For the case of a transverse staggered field a gap opens, which scales with $h_1^{\epsilon_1}$, where $\epsilon_1 = \epsilon_1(h_3)$ is given by the critical exponent $\eta_1(M(h_3))$ defined through the transverse structure factor of the model at $h_1 = 0$. For the case of a longitudinal periodic field with wave vector $q = \pi/2$ and strength h_q a plateau is found in the magnetization curve at $M = 1/4$. The difference of the upper- and lower magnetic field scales with $h_3^u - h_3^l \sim h_q^{\epsilon_3}$, where $\epsilon_3 = \epsilon_3(h_3)$ is given by the critical exponent $\eta_3(M(h_3))$ defined through the longitudinal structure factor of the model at $h_q = 0$.

PACS. 75.10 -b General theory and models of magnetic ordering

1 Introduction

The properties of the one-dimensional (1D) spin-1/2 antiferromagnetic Heisenberg model (AFH) with nearest neighbour coupling:

$$\mathbf{H}(h_3) \equiv \mathbf{H}_0 - 2h_3\mathbf{S}_3(0), \tag{1}$$

$$\mathbf{H}_0 \equiv 2\sum_{l=1}^{N} \mathbf{S}_l \cdot \mathbf{S}_{l+1}, \tag{2}$$

$$\mathbf{S}_a(q) \equiv \sum_{l=1}^{N} e^{ilq} S_l^a, \quad a = 1,2,3, \tag{3}$$

in the presence of a homogeneous external field of strength h_3 are well-known.

1. There is no gap. The magnetization curve $M = M(h_3)$ is a monotonically increasing convex function [1–3] for $h_3 \geq 0$; in particular there is no plateau.
2. In the presence of the field h_3 the ground state $|p_s, S\rangle$ of $\mathbf{H}(h_3)$ has total spin $S = S_T^3 = NM(h_3)$ and momentum $p_s = 0, \pi$ – depending on S and N.
3. The low energy excitations which can be reached from the ground state $|p_s, S\rangle$ by means of the transition operators $\mathbf{S}_3(q)$ and $\mathbf{S}_\pm(q)$:

$$\omega_3(q, h_3) = E(p_s + q, S) - E(p_s, S), \tag{4}$$

$$\omega_\pm(q, h_3) = E(p_s + q, S \pm 1) - E(p_s, S) \pm h_3, \tag{5}$$

vanish at the *soft mode* momenta $q_a = q_a(M)$:

$$\hat{\Omega}_a(M) \equiv \lim_{N\to\infty} N\omega_a(q_a(M), h_3), \tag{6}$$

with

$$q_a(M) = \pi \begin{cases} 1: & a = 1,2 \\ 1-2M: & a = 3 \end{cases}. \tag{7}$$

Conformal field theory describes the critical behaviour at the soft modes [4–8]. In particular the field dependence of the η-exponents:

$$\eta_a(M) = \frac{\hat{\Omega}_a(M)}{\pi v(M)} \tag{8}$$

has been computed by means of the Bethe ansatz solutions for the energy differences and the spin wave velocity [9,10]

$$v(M) = \frac{1}{2\pi}\lim_{N\to\infty} N[E(p_s + 2\pi/N, S) - E(p_s, S)]. \tag{9}$$

4. The η-exponents govern the finite-size behaviour of the transition amplitudes:

$$\langle S \pm 1, p_s + \pi|\mathbf{S}_\pm(\pi)|S, p_s\rangle \overset{N\to\infty}{\longrightarrow} N^{\kappa_1(h_3)} \tag{10}$$

$$\langle S, p_s + q_3|\mathbf{S}_3(q_3)|S, p_s\rangle \overset{N\to\infty}{\longrightarrow} N^{\kappa_3(h_3)} \tag{11}$$

with

$$\kappa_a(h_3) = 1 - \frac{\eta_a(M(h_3))}{2}, \tag{12}$$

and of the static structure factors:

$$\langle S, p_s|\mathbf{S}_a(q_a)\mathbf{S}_a(q_a)^\dagger|S, p_s\rangle \overset{N\to\infty}{\longrightarrow} N^{2-\eta_a(M)}. \tag{13}$$

At the soft mode momenta $q_a = q_a(M)$ the dynamical structure factors develop infrared singularities of the type $\omega^{-2\kappa_a(h_3)}$.

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First evidence for the existence of *low energy modes* in the excitation spectrum has been found recently in neutron scattering experiments on copper benzoat [11,12], exposed to a homogeneous magnetic field h_3. An exponential fit to the temperature dependence of the specific heat data revealed, however, that there is a gap in the energy differences (4, 5, 7), which opens with the field strength h_3 as h_3^ϵ, $\epsilon = 2/3$. This means of course, that the compound copper benzoat can not be described by a 1D Heisenberg antiferromagnet. Oshikawa and Affleck [13] argued that the local g-tensor for the Cu ions generates an effective staggered field of strength ($h_1 \ll h_3$), perpendicular to the uniform field h_3. Therefore, one is lead to investigate the Hamiltonian:

$$\mathbf{H}(h_3, h_1) \equiv \mathbf{H}(h_3) + 2h_1\mathbf{S}_1(\pi). \tag{14}$$

It is the purpose of this paper to study the evolution of the gaps

$$\omega_a(q_a, h_3, h_1) \propto h_1^{\epsilon_a(h_3)}, \tag{15}$$

by switching on the transverse staggered field. In particular we are interested in the h_3-dependence of the exponents $\epsilon_a(h_3)$.

It has been pointed out by the authors of reference [13] that a staggered field alone, *i.e.* $h_3 = 0, M = 0$, generates a ground state gap which opens with h_1^ϵ, $\epsilon = 2/3$. In a previous paper we have studied the finite-size scaling behaviour of the gap and of the staggered magnetization in the scaling limit $h_1 \to 0$, $N \to \infty$ and fixed scaling variable $x = Nh_1^\epsilon$ at $M = 0$.

The method used in reference [9] is based on a closed set of differential equations, which describes the h_1-evolution of the energy gap $\omega_3(\pi, 0, h_1)$ (Eq. (4)) and of the relevant transition amplitude (11) for $h_3 = 0$. It turns out that the exponent $\epsilon(h_3 = 0)$ in (15) is fixed by the finite-size behaviour of the initial values, *i.e.* (4, 11) for $h_3 = h_1 = 0$:

$$\epsilon(h_3 = 0) = \frac{1}{1 + \kappa(0)} = \frac{2}{3}\,. \tag{16}$$

In this paper, we extend the method of reference [9] to the case $h_3 > 0$.

In Section 2 we discuss the evolution equations for the Hamiltonian (14). The finite-size behaviour of the initial conditions ($h_1 = 0, h_3 > 0$) for the gaps (4, 5) and for the relevant transition matrix elements (10, 11) is reviewed as well.

Switching on the transverse staggered field in (14) a gap opens at the field independent (Eq. (5) for $q = \pi$) and the field dependent (Eq. (4) for $q = q_3(M)$) soft modes. The finite-size scaling behaviour of these gaps is studied in Sections 2.1 and 2.2, respectively. In Section 3, we investigate the effect of a longitudinal periodic field on the low-energy excitations of the AFH model. From these results we infer in Section 3.1 the corresponding magnetization curve.

2 Evolution equation and initial conditions

Starting from the eigenvalue equation of the Hamiltonian (14)

$$\mathbf{H}(h_3, h_1)|\Psi_n(h_3, h_1)\rangle = E_n(h_3, h_1)|\Psi_n(h_3, h_1)\rangle, \tag{17}$$

it is straight forward to derive the following set of differential equations

$$\frac{d^2E_n}{dh_1^2} = -2\sum_{l\neq n}\frac{|T_{ln}|^2}{\omega_{ln}}, \tag{18}$$

$$\frac{dT_{nm}}{dh_1} = -\sum_{l\neq m,n}\left[\frac{T_{nl}T_{lm}}{\omega_{ln}} + \frac{T_{nl}T_{lm}}{\omega_{lm}}\right] - \frac{T_{nm}}{\omega_{nm}}\frac{d\omega_{nm}}{dh_1}, \tag{19}$$

which describes the evolution of the energy eigenvalues $E_n = E_n(h_3, h_1)$, energy differences $\omega_{nm} = \omega_{nm}(h_3, h_1) = E_n - E_m$ and transition matrix elements

$$T_{nm}(h_3, h_1) \equiv \langle\Psi_n(h_3, h_1)|\mathbf{S}_1(\pi)|\Psi_m(h_3, h_1)\rangle, \tag{20}$$

of the perturbation operator $\mathbf{S}_1(\pi)$[1]. The latter has the following properties: it changes the momentum by $\Delta p = \pi$ and the total spin S_T^3 by one unit. Therefore, the eigenstates $|\Psi_n(h_3, h_1)\rangle$ are linear combinations

$$|\Psi_n(h_3, h_1)\rangle = \sum_{S_T^3}\big[a_n(S_T^3, h_1)|p_n, S_T^3\rangle + b_n(S_T^3, h_1)|p_n + \pi, S_T^3\rangle\big]\,, \tag{21}$$

of eigenstates $|p_n, S_T^3\rangle$ and $|p_n + \pi, S_T^3\rangle$ to the total spin S_T^3 and the momenta $p_n, p_n + \pi$. Note, that the evolution equations (18, 19) decouple for different momenta p_n, p_m with $|p_n - p_m| \neq \pi$. In Sections 2.1 and 2.2 we will study the following cases:

1. $p_n = 0, \pi$,
2. $p_n = q_3(M),\ q_3(M) + \pi$.

For both cases we have the initial conditions:

$$\omega_{nm}(q, h_3, h_1 = 0) = \frac{a_{nm}(h_3)}{N}, \tag{22}$$

$$T_{nm}(h_3, h_1 = 0) = b_{nm}(h_3)N^{\kappa(h_3)}, \tag{23}$$

which are completely fixed by the excitation energies and transition amplitudes of the unperturbed problem ($h_1 = 0$) in a uniform field h_3. We can now repeat the whole line of arguments, we developed for $h_3 = 0$ in reference [9]. The evolution equations (18, 19) possess scaling solutions:

$$\omega_{nm}(q, h_3, h_1) = h_1^{\epsilon(h_3)}\Omega_{nm}(x), \tag{24}$$

$$T_{nm}(h_3, h_1) = Nh_1^{\sigma(h_3)}\Theta_{nm}(x), \tag{25}$$

in the combined limit

$$h_1 \to 0, \quad N \to \infty, \quad x \equiv Nh_1^{\epsilon(h_3)} \text{ fixed.} \tag{26}$$

[1] The N-dependence of eigenvalues and transition matrix elements is always understood.

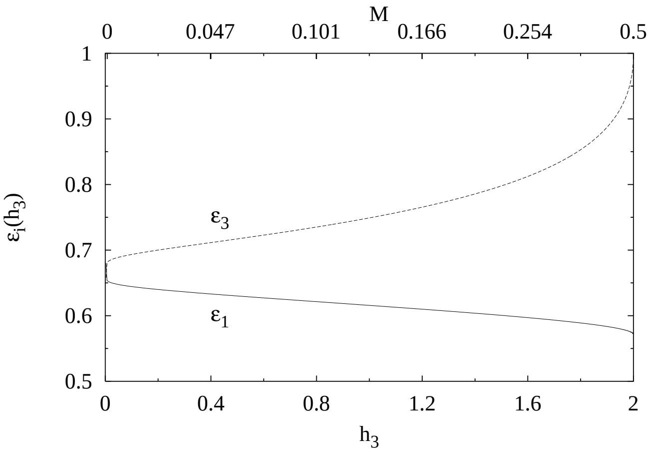

Fig. 1. The exact critical exponents ϵ_1 (solid line) and ϵ_3 (dashed line) *versus* h_3 and M, determined from a Bethe ansatz solution of finite system size $N = 4096$.

The exponents $\epsilon(h_3)$ and $\sigma(h_3)$ are given by the finite-size behaviour of the initial values (22, 23):

$$\epsilon(h_3) = \frac{1}{1+\kappa(h_3)}, \quad \sigma(h_3) = \frac{1-\kappa(h_3)}{1+\kappa(h_3)} \cdot \tag{27}$$

2.1 The gap at the field independent soft mode $\mathbf{q} = \pi$

As was pointed out in the introduction, the ground state $|n=0\rangle = |p_s, S\rangle$ of the 1D spin-1/2 AFH model, $\mathbf{H}(h_3, 0)$, in the presence of a uniform field h_3 has total spin $S_T^3 = S = NM(h_3)$ and momentum $p_s = 0$, or $p_s = \pi$. The first excited state which can be reached with the operator $\mathbf{S}_1(\pi)$:

$$|n = \pm 1\rangle = |p_s + \pi, S_T^3 = S \pm 1\rangle, \tag{28}$$

has a gap of the type (22)

$$\omega_{\pm 10}(\pi, h_3, 0) = E(p_s + \pi, S \pm 1) - E(p_s, S) \mp h_3, \tag{29}$$

which vanishes as N^{-1} for $N \to \infty$. The transition matrix elements:

$$T_{\pm 10}(h_3, 0) \equiv \langle \pm 1 | \mathbf{S}_\pm(\pi) | 0 \rangle \stackrel{N\to\infty}{\longrightarrow} N^{\kappa_1(h_3)} \tag{30}$$

diverge in the limit $N \to \infty$, where $\kappa_1(h_3)$ is obtained from the known $\eta_1(M)$ exponent (12). Both curves, $\eta_1 = \eta_1(M)$ and $M = M(h_3)$ were determined exactly by means of Bethe ansatz solutions on large systems [8], as well *via* a solution of a system of nonlinear integral equation derived from the Bethe ansatz [10]. The h_3-dependence is shown in Figure 1. It starts at the known value $\epsilon_1(h_3 = 0) = 2/3$ and then drops monotonically with h_3. At $h_3(M = 1/4) = 1.58\ldots$, the exponent is reduced to

$$\epsilon_1(h_3(1/4)) = 0.5975\ldots \tag{31}$$

In order to explore the scaling behaviour (24) of the gap, we have determined numerically the ratios:

$$\begin{aligned}\frac{\omega_{10}(\pi, h_3, h_1)}{\omega_{10}(\pi, h_3, 0)} &= 1 + e_{10}(x, h_3), \\ &= 1 + x\frac{\Omega_{10}(x, h_3)}{a_{10}},\end{aligned} \tag{32}$$

with $x = Nh_1^{\epsilon_1(h_3)}$ and Ω_{10} as given in reference [9], on finite systems. The homogeneous field h_3 has to be chosen carefully. According to our premise, the ground state $|0\rangle = |p_s, S\rangle$ (at $h_1 = 0$) has total spin $S_T^3 = S = NM(h_3)$ and energy $E(p_s, S) - 2h_3 S_T^z$. The two excited states $|\pm 1\rangle = |p_s{+}\pi, S_T^3 = S{\pm}1\rangle$ have a gap. Positivity of the gaps yields an upper and lower bound of the h_3-field ($h_3^u \geq h_3 \geq h_3^l$):

$$2h_3^u = E(p_s{+}\pi, S{+}1) - E(p_s, S), \tag{33}$$

$$2h_3^l = E(p_s, S) - E(p_s{+}\pi, S{-}1), \tag{34}$$

which leads to the well-known steps in the magnetization curve on finite systems [1]. Note, that at the edges h_3^u and h_3^l the excitations energies:

$$\omega_{+10}(\pi, h_3^u, h_1 = 0) = 0, \tag{35}$$

$$\omega_{-10}(\pi, h_3^l, h_1 = 0) = 0, \tag{36}$$

vanish identically. Therefore, ratios of the gap (32) do not make sense in these cases. At the midpoint field $\bar{h}_3$, however:

$$\begin{aligned}2\bar{h}_3 &\equiv (h_3^u + h_3^l)/2 \\ &= [E(p_s{+}\pi, S{+}1) - E(p_s{+}\pi, S{-}1)]/2,\end{aligned} \tag{37}$$

the two excited states have the same gap:

$$\omega_{\pm 10}(\pi, \bar{h}_3, 0) = \frac{E(p_s{+}\pi, S{+}1) + E(p_s{+}\pi, S{-}1) - 2E(p_s, S)}{2} \cdot \tag{38}$$

The degeneracy of these two excited states is not lifted in the first oder perturbation theory in h_1, since all the relevant matrix elements

$$\langle n | \mathbf{S}_1(\pi) | m \rangle = 0, \quad n, m = \pm 1 \tag{39}$$

vanish. The ratio (32) is shown in Figure 2a, for the midpoint field $\bar{h}_3 = \bar{h}_3(N) \approx 1.58$, corresponding to a magnetization $M = 1/4$ on system sizes $N = 8, 12, 16, 20$. Optimal scaling is achieved here, with the exponent $\epsilon_1 = 0.595(5)$, which is in excellent agreement with the exact value (31). According to reference [9], the low x-behaviour of the scaling function $e_{10}(x, h_3)$ is also predicted by the evolution equations (18, 19) in the scaling limit (26):

$$e_{10}(x, h_3) = e_{10}(h_3) x^{\phi_1(h_3)}, \tag{40}$$

with $\phi_1(h_3) = 2/\epsilon_1(h_3)$. The linear behaviour in the variable $x^{2/\epsilon_1(h_3)}$ for small x-values is clearly seen in Figure 2a.

The effect of the homogeneous h_3-field on the exponent ϵ_1 is demonstrated in Figure 2a. An exponent $\epsilon_1(h_3) = \epsilon_1(h_3 = 0) = 2/3$ independent of h_3 would lead to considerable scaling violations of the ratios (32), as is demonstrated in Figure 2b.

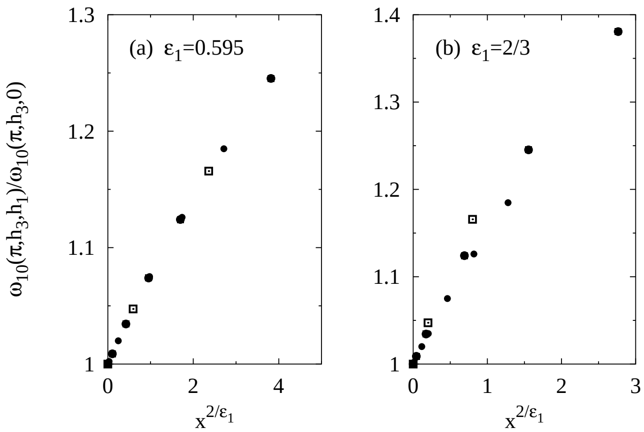

Fig. 2. A comparison of the ratio (32) for two different values of ϵ_1 and the midpoint field $\bar{h}_3$ (Eq. (37)) for system sizes $N = 8, 12, 16, 20$.

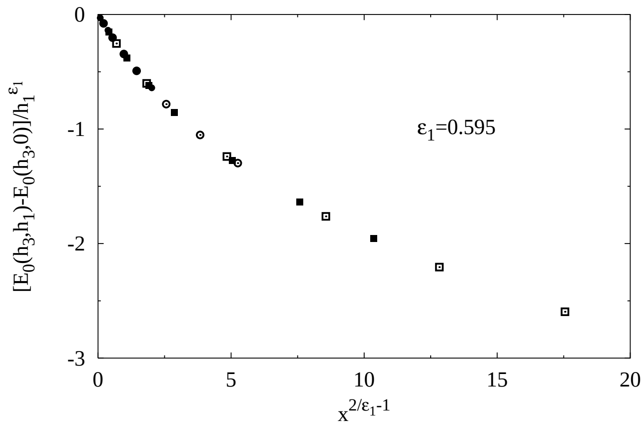

Fig. 3. The scaling of the ground state energy (45) for the midpoint field $\bar{h}_3$ (Eq. (44)) in the limit (26). Numerical data were obtained on system sizes $N = 8, 12, 16, 20$.

2.2 The gap at the field dependent soft mode $\mathbf{q = q_3}$

Let us now turn to the field dependent soft mode (Eq. (4) for $q = q_3(M)$). Switching on the perturbation operator $h_1\mathbf{S}_1(\pi)$ the ground state energy $E(p_s, S)$ and the energy $E(p_s{+}q_3(M), S)$ of the excited state evolve independently, since their momentum difference $q_3(M)$ does not fit to the momentum transfer π mediated by the operator $\mathbf{S}_1(\pi)$. In other words, we have to study the ground state energy $E_0(h_3, h_1)$ in the sectors with momentum $p = 0, \pi$ and $p = q_3(M), q_3(M) + \pi$, separately. In both cases insertion of the scaling ansatz (24, 25) for the excitation energies ω_{n0} and transition amplitudes T_{n0} into (18) yields:

$$\frac{d^2E_0}{dh_1^2} = -N^{1+2\kappa_1(h_3)} x^{1-2\kappa_1(h_3)} \sum_{l\neq 0} \frac{|\Theta_{l0}(x)|^2}{\Omega_{l0}(x)}, \qquad (41)$$

where $x = Nh_1^{\epsilon_1(h_3)}$. To integrate (41) we introduce

$$y \equiv x^{1/\epsilon_1(h_3)} = h_1 N^{1+\kappa_1(h_3)}, \qquad (42)$$

and

$$f(y) \equiv x^{1-2\kappa_1(h_3)} \sum_{l\neq 0} \frac{|\Theta_{l0}(x)|^2}{\Omega_{l0}(x)}, \qquad (43)$$

from which follows:

$$E_0(h_3, h_1) - E_0(h_3, 0) = - \left(\frac{h_1}{y}\right)^{\epsilon_1(h_3)} \int_0^y dy' \int_0^{y'} dy'' f(y''). \qquad (44)$$

Here, we have used the fact that

$$\left.\frac{dE_0(h_3, h_1)}{dh_1}\right|_{h_1=0} = \langle 0|\mathbf{S}_1(\pi)|0\rangle|_{h_1=0} = 0. \qquad (45)$$

Equation (44) describes the lowering of the ground state energy, if we switch on the staggered field of strength h_1. We observe the same scaling behaviour with $h_1^{\epsilon_1}$, we found for the excitation energies $\omega_{n0}(\pi, h_3, h_1)$. In Figure 3 we have plotted $[E_0(h_3, h_1) - E_0(h_3, 0)]/h_1^{\epsilon_1(h_3)}$ *versus* the scaling variable $x^{2/\epsilon_1 - 1}$ for the case $p = 0, \pi$. We observe a linear behaviour in this variable, which is a consequence of the small x-behaviour of the energy differences $\Omega_{l0}(x)$ and transition amplitudes $\Theta_{l0}(x)$ in equation (43) (see Ref. [9]):

$$\Omega_{l0}(x) \sim \frac{a_{l0}}{x}, \quad \Theta_{l0}(x) \sim x^{-2+1/\epsilon_1}. \qquad (46)$$

Therefore, the integrand on the right-hand side of (44) is constant and the small x-behaviour of (44) is governed by $y^{2-\epsilon_1} = x^{2/\epsilon_1 - 1}$.

Let us next turn to the lowering of the ground state energy in the sector with $p = q_3(M), q_3(M){+}\pi$. The exponents $\kappa_\pm(h_3)$ are defined by the initial conditions ($h_1 = 0$) for the transition matrix elements:

$$\begin{aligned}\langle \pm 1|\mathbf{S}_1(\pi)|0\rangle &= \langle p_{s\pm 1}{+}q_3(M), S{\pm}1|\mathbf{S}_1(\pi)|p_s{+}q_3(M), S\rangle \\ &= b_{\pm 10}(h_3)N^{\kappa_\pm(h_3)}.\end{aligned} \qquad (47)$$

Conformal field theory relates the corresponding η-exponents ($\kappa_\pm = 1 - \eta_\pm/2$) to the scaled energy differences:

$$\eta_\pm(M) = \frac{\hat{\Omega}_\pm(M)}{\pi v(M)}, \qquad (48)$$

with

$$\begin{aligned}\hat{\Omega}_\pm(M) = \lim_{N\to\infty} N\Big[&E(p_{s\pm 1}{+}q_3(M), S{\pm}1) \\ &-E(p_s{+}q_3(M), S)\Big].\end{aligned} \qquad (49)$$

Here $v(M)$ is the spin wave velocity (9) at the soft mode $q = 0$. Evaluating (49, 9) leads to the following representation of the $\eta_\pm$-exponents (48)

$$\eta_\pm(M) = \eta_1(M) + \frac{v_\pm(M)}{v(M)}, \qquad (50)$$

where

$$v_\pm(M) = \frac{1}{2\pi}\lim_{N\to\infty} N\Big[E(p_{s\pm1}+q_3(M\pm1/N)\pm2\pi/N, S\pm1) \\ -E(p_{s\pm1}+q_3(M\pm1/N), S\pm1)\Big], \quad (51)$$

are the right-hand- (+) and left-hand (−) spin wave velocities obtained from the slopes of the dispersion curve approaching the soft mode momentum from the right- and from the left-hand side, respectively:

$$p \mapsto p_s + q_3(M) \pm 2\pi/N. \quad (52)$$

From conformal invariance arguments for the energy differences in (51) we get

$$\eta_+(M) = \eta_-(M) = 1 + \eta_1(M). \quad (53)$$

In summary, we conclude that the gap of the field dependent soft mode $q_3(M)$:

$$E(p_s+q_3(M), S) - E(p_s, S) \sim h_1^{\epsilon_1(h_3)}, \quad (54)$$

is dominated by the lowering of ground state energy $E(p_s, S)$ and therefore scales with the same exponent $\epsilon_1(h_3)$ as the field independent one.

3 Opening of a gap in a longitudinal periodic field

So far we have only considered the Hamiltonian (14) with an inhomogeneous field $h_1\mathbf{S}_1(\pi)$ transverse to the homogeneous field $h_3\mathbf{S}_3(0)$. By means of the evolution equations (18, 19) we can also study the influence of a longitudinal periodic field

$$\mathbf{H}(h_3, h_q) \equiv \mathbf{H}_0 - 2h_3\mathbf{S}_3(0) + 2h_q\bar{\mathbf{S}}_3(q). \quad (55)$$

The perturbation operator $\bar{\mathbf{S}}_3(q) \equiv [\mathbf{S}_3(q) + \mathbf{S}_3(-q)]/2$ commutes with the total spin operator S_T^3 and changes the ground state momentum p_s by $\pm q$. For this reason, all momentum states with

$$p_k = p_s \pm kq, \quad k = 0, \pm1, \pm2, \ldots \quad (56)$$

are coupled *via* the evolution equation. For example for $q = \pi/2$ there are 4 different momentum states with $p_k/\pi = \pm1/2, 0, 1$, which have to be taken into account. In general, the transition matrix elements at $h_q = 0$:

$$T_3(h_3, h_q = 0) = \langle p_s \pm q, S|\mathbf{S}_3(\pm q)|p_s, S\rangle \quad (57)$$

turn out to be finite, except for the case, where we meet a soft mode:

$$\omega_3(q, h_3, h_q = 0) = E(p_s+q, S, h_q = 0) - E(p_s, S, h_q = 0) \\ \overset{N\to\infty}{\longrightarrow} \frac{a_3(h_3)}{N}. \quad (58)$$

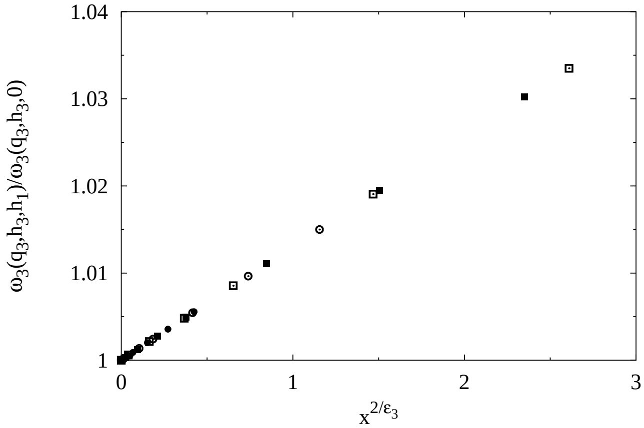

Fig. 4. Finite-size scaling of the gap ratio (61), for $N = 8, 12, 16, 20$ and $q_3(M = 1/4) = \pi/2$ with $\epsilon_3 = 0.81\ldots$

This happens if:

$$q = q_3(M) = \pi(1 - 2M), \quad (59)$$

e.g. a soft mode appears at $q = \pm\pi/2$ if $M = 1/4$. At the soft mode (59) the transition matrix elements (57) diverge:

$$T_3(h_3, 0) \overset{N\to\infty}{\longrightarrow} b_3(h_3)N^{\kappa_3(h_3)} \quad (60)$$

with an exponent $\kappa_3(h_3) = 1 - \eta_3(M(h_3))/2$, given by the $\eta_3(M)$-exponent, given in the introduction. From the evolution equations with the initial conditions (58, 60), we get in this case a finite-size scaling behaviour of the gap ratio:

$$\frac{\omega_3(q_3, h_3, h_q)}{\omega_3(q_3, h_3, 0)} = 1 + e_3(x, h_3), \quad (61)$$

with a scaling variable $x = Nh_q^{\epsilon_3(h_3)}$, where $\epsilon_3(h_3) = 1/[1 + \kappa_3(h_3)]$. The curve $\epsilon_3(h_3)$ is shown in Figure 1. Note that $\eta_3(M) = 1/\eta_1(M)$, which means $\epsilon_3(0) = 2/3$, *e.g.* for $M = 1/4$, we have

$$\epsilon_3(h_3(M = 1/4)) = 0.81011\ldots \quad (62)$$

A test of the finite-size scaling behaviour (61) for $q = \pi/2$ and $M = 1/4$ with the exponent (62) is shown in Figure 4. The small x-behaviour of the gap ratio is properly reproduced with x^{2/ϵ_3} and compared with the prediction $h_q^{\epsilon_3}$, where $\epsilon_3 = \epsilon_3(h_3(M = 1/4))$ is given by (62).

3.1 The magnetization curve in a periodic field

Let us finally discuss the influence of the periodic perturbation in (55) on the magnetization curve $M = M(h_3)$. First of all one should notice that the opening of a gap for $h_q > 0$ in the energy differences (58) does not imply a priori a plateau in the magnetization curve. The criterium

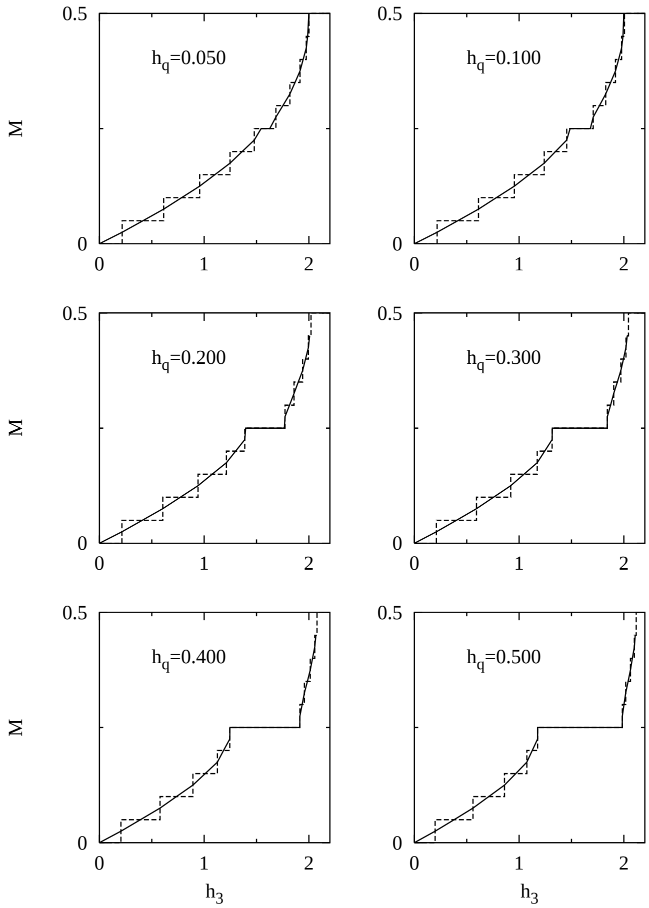

Fig. 5. The evolution of a plateau in the magnetization curve at $M = 1/4$, induced by an external field (55) with period $q = \pi/2$. The magnetization curve is calculated from finite system ($N = 20$) *via* midpoint magnetization [1] in conjunction with a finite-size extrapolation of the plateau width from system sizes of $N = 8, 12, 16, 20$.

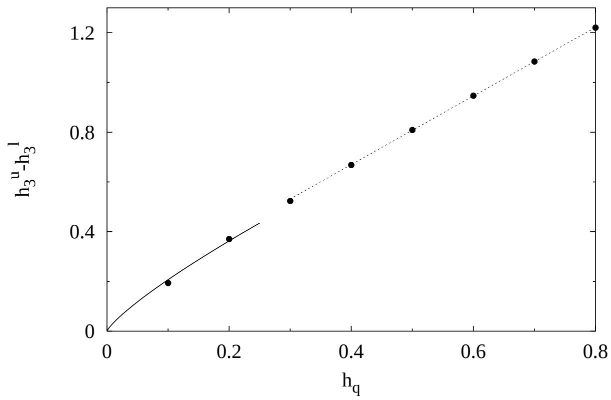

Fig. 6. The evolution of the difference (63) between the upper and lower critical field at the plateau $M = 1/4$. The solid line shows a fit to the data for small values of the external periodic field h_q. The expected behaviour is $\propto h_q^{\epsilon_3}$, with $\epsilon_3 = 0.8101$ given by equation (62). The dashed line represents a linear fit for larger values of h_q.

of a plateau with an upper and lower critical field h_3^u, h_3^l can be read from (33, 34):

$$2(h_3^u - h_3^l) = \lim_{N\to\infty}\Big[E(p_s+\pi, S+1, h_q) -2E(p_s, S, h_q)E(p_s+\pi, S-1, h_q)\Big]. \quad (63)$$

The emergence of the plateaus in the magnetization curve can be seen in Figure 5. A finite-size analysis shows that a non vanishing difference (63) remains in the thermodynamical limit. For this analysis we have used the BST-Algorithm [14,7]. The h_q-dependence of the plateau width is plotted in Figure 6, together with the predicted scaling behaviour $h_q^{\epsilon_1}$ for $q = \pi/2$.

4 Conclusions

This paper is aimed to study the effect of a small periodic field on the eigenvalue spectrum of the 1D spin-1/2 AFH model. We are interested in particular in the opening of a gap in those situations, where the unperturbed model is known to be critical. The critical exponents $\eta_1(M), \eta_3(M)$, which govern the divergence in the transition matrix elements (10, 11) of the unperturbed model, are known. Following conformal field theory, they are related to the finite-size behaviour (6) of certain energy differences (4, 5), which can be computed on very large systems by means of Bethe ansatz.

The evolution of the eigenvalue spectrum under the influence of perturbation of strength h_q is described by a system of differential equations (18, 19), which has been shown to have scaling solutions (24, 25) in the scaling limit (26). The exponents ϵ and σ in the scaling solutions are uniquely determined by the corresponding η-exponents in the unperturbed model. We have studied in detail the following types of perturbations.

1. A transverse staggered field together with a homogeneous longitudinal field $h_1\mathbf{S}_1(\pi) + h_3\mathbf{S}_3(0)$. Both energy differences (4, 5) at the soft mode momenta (7) were shown to evolve a gap with an exponent

$$\epsilon_a(h_3) = \frac{2}{4 - \eta_a(M(h_3))}, \quad (64)$$

   with $a = 1$ depending on the external homogeneous field h_3 with magnetization $M(h_3)$.
2. A longitudinal homogeneous and periodic field $2h_3\mathbf{S}_3(0) + 2h_q\bar{\mathbf{S}}_3(q)$. Such a perturbation creates a plateau in the magnetization curve $M = M(h_3)$ at

$$M = \frac{1}{2}\left(1 - \frac{q}{\pi}\right). \quad (65)$$

   In other words q has to meet the soft mode momentum $q = q_3(M) = \pi(1 - 2M)$. The difference of the upper and lower critical field, which defines the width of the plateau, evolves with an exponent $\epsilon_3(h_3)$, which is related to the corresponding η_3-exponent *via* (64) for $a = 3$.

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